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Asymptotic Behavior of Planetary-Scale Motions of the Atmosphere

G. K. Morikawa

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ABSTRACT

Asymptotic equations describing slowly varying planetary-scale motions of the atmosphere over the rotating earth are derived and compared to the geostrophic equations of motion in the tangent plane approximation. Since the same approximation procedure is used in both the spherical and plane geometry, a direct term by term comparison is possible. The plane approximation yields familiar prognostic equations (initial value problem); but in the more constrained spherical geometry, a more restricted forcing-type problem results if the zonal motion is time-dependent. That is, with a given slowly varying or steady zonal motion, is associated a class of long-wave tesseral motions, i.e. coupled low-wave-number and meridional motions.

1. Introduction

In recent years dynamic meteorologists have been attempting to obtain approximate equations of motion describing the large-scale, nearly-horizontal flow of the atmosphere valid over the entire rotating earth. These studies are a continuation of the pioneering work of C.-G. Rossby and have been greatly stimulated by the rapid development of large electronic computers. A review article by Phillips [1] indicates the present understanding about this problem.

In this paper we derive approximate equations describing adiabatic planetary-scale motions of the atmosphere valid for asymptotically large time from the viewpoint of the original fluid equations. Although this approximation procedure yields an ordered system of linear partial differential equations, the coefficients are variable; and numerical calculations must be carried out in the future in order to verify if the actual physical atmosphere allows such motions to exist.*

One of the principal results which we can expect to obtain from numerical studies of these asymptotic equations is the determination of the secondary motions associated with a slowly varying zonal motion. These secondary motions consist of a low-wave-number eddy motion closely coupled with a non-geostrophic meridional motion. The novelty of these equations is that they are apparently valid over the entire earth, including the poles and the equator where the usual geostrophic approximation formally breaks down. However the highly constrained nature of this rather degenerate system of equations seems to restrict the asymptotic motion to slowly-time-varying nearly-zonal flows combined with an associated meridional motion. In §3.1 we find that long-wave tesseral motions are eliminated by the approximation procedure when we perturb on the atmosphere at rest with respect to the rotating earth. But in §3.2 we show how the more complex tesseral motions, which are associated with a prescribed slowly varying zonal motion, can be studied.

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*What is implied by these asymptotic equations is that heating and dissipation (not included in the analysis) act mainly on smaller scale elements in the atmosphere and do not directly modify the large scale motions except the purely zonal flow itself which must have an external driving mechanism. This is only approximately true at best.

The mathematical technique used here is the same as that which yields the approximate first-order geostrophic motion in the tangent-plane approximation (Morikawa [2]). The formal procedure is the following:

1) Scale the time t with respect to a small parameter ε, where $\varepsilon << 1$; the new time variable is $\tau = \varepsilon t$. The small parameter ε may be related to the strength or amplitude of disturbances in the

initial or boundary data or to the frequency of the slowly varying motions.

2) Express all the dependent physical variables as power series expansions in ε, e.g. in §3.1, perturbed on the atmosphere at rest with respect to the rotating earth; the zeroth order terms describe the solid body rotation of a stable gravitating, compressible atmosphere, stationary with respect to the earth.

The time scaling (or "time stretching") performs the task of filtering out all but the slowly varying, large-scale motions which are directly related to the asymptotically large-time behavior of solutions of the original system of equations.

Now we make some conjectures: A possible implication of these asymptotic equations is that for some large time scale, the planetary meridional circulation is driven mainly by the zonal flow, but for smaller time (or length) scales the zonal flow is forced by the smaller scale motions, i.e. cyclonic and anti-cyclonic motions which have more detailed secondary flow structure, directly associated with them, than the larger cellular meridional motions. Our results here might be regarded as being complementary to both the tangent-plane and Rossby's β-plane approximations. That is, the present asymptotic analysis describes primarily some planetary motions in the long wave limit of zero wave number in §3.1 and low-wave number in §3.2; the plane approximations appear to have validity in some range of intermediate wave numbers between zero and infinity (say, between five and fifty). There is also the possibility that the actual atmosphere allows a related spacially complementary description within the framework of our asymptotic equations. That is, the planetary analysis may be approximately valid only near the equator and possibly near the poles; and the tangent plane description is approximately valid only in middle latitudes. A more comprehensive analysis, which includes a greater range of atmospheric phenomena and which includes both approximations as special cases, is needed to explain the precise relationship between them.* This may require a more general approximation procedure which allows the introduction of at least one additional time (or length) scale. The direction in which to search for such a formulation may lie in the study of large-scale atmosphic turbulence; then heating and dissipation must be at least implicit in the future analysis, which would be a difficult task indeed.

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*A thorough numerical study of the asymptotic equations described in §3.2 may furnish some relevant information about this relationship. However, the required global meteorological data, preferably daily over a period of at least one year, is apparently not yet available.

2. Conservation of Potential Vorticity and the Equations of Motion.

In the study of large-scale motions of the atmosphere in which the rotation of the earth plays an important role, the concept of potential vorticity advection (or convection) is vital. This is especially accentuated in adiabatic flows since, there, the potential vorticity is conserved on particle paths. In a frame of reference rotating with constant angular velocity $\vec{\Omega}$, the potential vorticity conservation theorem, which is attributed* to Euler (1757), Ertel (1942) and Rossby (1949), states

$$\text{(1)} \qquad \frac{d}{dt}\{(2\vec{\Omega} + \vec{\zeta}) \cdot (\rho^{-1} \operatorname{grad} \Theta)\} = 0$$

where d( )/dt is the three-dimensional particle (or material) derivative operator, $\vec{\zeta} = \operatorname{curl} \vec{v}$ is the relative vorticity, ρ is the density, and Θ is the potential temperature (directly related to the entropy $S = C_p \ln \Theta$, where C_p is the specific heat at constant pressure).

The derivation of Eq. (1) requires the explicit use of the following equations valid in the rotating reference frame:

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*In the more recent literature an exposition of these and related matters is given by Truesdell [3] in his treatise on vorticity; more directly on meteorological subjects, we refer to Eliassen and Kleinschmidt [5].

1) Conservation of mass (or continuity equation)

(2) $$\frac{d\rho}{dt} + \rho \operatorname{div} \vec{v} = 0$$

2) Vorticity advection equation obtained by operating with the curl ( ) on the momentum equation,

(3) $$\frac{d\vec{v}}{dt} + 2\vec{\Omega} \times \vec{v} + \rho^{-1} \operatorname{grad} p + \operatorname{grad} \Phi = 0$$

where

(4) $$\frac{d\vec{v}}{dt} \equiv \vec{v}_t + (v \cdot \operatorname{grad})\vec{v} \equiv \vec{v}_t - \vec{v} \times \operatorname{curl} \vec{v} + \frac{1}{2} \operatorname{grad} |\vec{v}|^2$$

and p is the fluid pressure, Φ is the gravitational potential, and subscripts indicate partial differentiation; the resulting vorticity advection equation is

(5) $$\frac{d\vec{\zeta}}{dt} + (2\vec{\Omega} + \vec{\zeta}) \operatorname{div} \vec{v} - [(2\vec{\Omega} + \vec{\zeta}) \cdot \operatorname{grad}]\vec{v} + (\operatorname{grad} \rho^{-1}) (\operatorname{grad} p) = 0$$

3) Conservation of potential temperature for adiabatic flow,

(6) $$\frac{d\Theta}{dt} = 0$$

where the equation of state relates Θ, p and ρ,

(7) $$\Theta = (\gamma p_0^{\frac{\gamma-1}{\gamma}} \cdot p^{\frac{1}{\gamma}}) / \rho$$

where p_0 is the pressure at the ground, r = a, and the gas constant $\gamma = C_p/C_v$, the ratio of specific heats.

For our basic complete system of equations we choose Eqs. (1), (3), (6) and (7) and the appropriate initial and boundary conditions on the surface of the earth and at infinity. A completely equivalent system of equations is given by replacing the potential vorticity equation (1) by the continuity equation (2); however, the asymptotic behavior of the motion is obtained more succinctly by suppressing the explicit use of Eq. (2) (cf. [2]).

3. Asymptotic Equations of Large-Scale Planetary Motions

The formal approximation procedure which we use to obtain the asymptotic equations is quite simple. That is, the rules are elementary although the complete physical and mathematical justification and precise estimates of error involved in the approximation are not totally apparent at the present time. Previously we have used this same procedure to obtain approximate equations describing two-dimensional geostrophic vortex motion on a plane tangent to the rotating earth (Ref. [2]). In Appendix I, we derive the equivalent geostrophic equations of motion[*] for an adiabatic, compressible atmosphere over a tangent plane, i.e. in which the horizontal and vertical components of the earth's rotation rate vector $\vec{\Omega}$ is assumed to be constant and fixed to the latitude at the point of tangency. In Appendix II we show the relationship between the vertical structure of the atmosphere (at rest with respect to the tangent plane) and the vertical variation of the horizontal length scale of a singular geostrophic vortex line element.[**] At the present time the motivation for, and validity of, the asymptotic equations must rest partly on the successful derivation of approximate equations for geostrophic-motion over the tangent plane, as derived in these Appendices. In addition, further substantiation must await the testing of these asymptotic equations by comparison with actual atmospheric motions, using numerical methods aided by electronic computers.

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[*] Apparently these equations are similar to those first derived by Obukhov [5] and Monin [6] by a related method.

[**] In addition, the strength of the vortex is functionally related to the vertical structure of the static atmosphere and hence to the horizontal length scale.

3.1. Asymptotic Equations Obtained by Perturbing on the Atmosphere at Rest.

In order to proceed with the derivation we first transform the basic equations (1), (3), and (6), which are in vector form, into equations in spherical polar coordinates and introduce the scaled time $\tau = \varepsilon t$. Then the independent variables are (θ, ϕ, r, τ) where θ is longitude increasing eastward, ϕ is latitude increasing northward from the equator, and r is radius measured from the earth's center. The velocity components corresponding to the coordinates (θ, ϕ, r) are $(u, v, w) \equiv \vec{v}$. Thus the dependent variables expressed as a state vector $(u, v, w, \rho, p, \Theta)$ for the atmosphere at rest $(\varepsilon = 0)$ is $(0, 0, 0, \rho^{(0)}, p^{(0)}, \Theta^{(0)})$ representing the lowest order terms in the power series expansion with respect to ε and is independent of time. The potential vorticity equation

(1) is transformed into spherical polar coordinates using the following identities: The earth's rotation vector becomes

$$(8) \qquad 2\vec{\Omega} = (0,\ 2\Omega\cos\phi,\ 2\Omega\sin\phi)$$

and we abbreviate by calling the ϕ-component, $2\Omega\cos\phi = f_2$, and the r-component, $2\Omega\sin\phi = f_3$; the relative vorticity is

$$(9) \qquad \vec{\zeta} = \text{curl}\ \vec{v} = (\xi,\ \eta,\ \zeta)$$

$$= \left\{\frac{1}{r}\left[w_\phi - (rv)_r\right],\ \frac{1}{r}\left[(ru)_r - \frac{w_\theta}{\cos\phi}\right],\ \frac{1}{r\cos\phi}\left[v_\theta - (u\cos\phi)_\phi\right]\right\};$$

the gradient of the potential temperature is

$$(10) \qquad \text{grad}\ \Theta = \left(\frac{\Theta_\theta}{r\cos\phi},\ \frac{\Theta_\phi}{r},\ \Theta_r\right)\ ;$$

and the particle derivative operator on a scalar quantity is

$$(11)\ \frac{d(\ )}{dt} = \varepsilon(\ )_\tau + \frac{u}{r\cos\phi}(\ )_\theta + \frac{v}{r}(\ )_\phi + w(\ )_r$$

where lettered subscripts denote partial differentiation. Also the potential temperature equation (6) is immediately transformed into spherical polar coordinates by Eq. (11). The $(\theta,\ \phi,\ r)$ components of the momentum equation (3) are

$$(12)\quad \frac{du}{dt} + \frac{uw}{r} - \frac{uv\ \tan\phi}{r} + \frac{p_\Theta}{\rho r \cos\phi} + 2\Omega\,(w \cos\phi - v \sin\phi) = 0$$

$$(13)\quad \frac{dv}{dt} + \frac{vw}{r} + \frac{u^2 \tan\phi}{r} + \frac{p_\phi}{\rho r} + 2\Omega\, u \sin\phi = 0$$

$$(14)\quad \frac{dw}{dt} - \frac{(u^2+v^2)}{r} + \frac{p_r}{\rho} + \frac{a^2 g}{r^2} - 2\Omega\, u \cos\phi = 0$$

where a is the earth's radius and g is the gravitational acceleration at $r = a$. For completeness, div $\vec{v}$ becomes

$$(15)\quad \operatorname{div} \vec{v} = \operatorname{div}\,(u,\ v,\ w) = \frac{u_\Theta}{r\cos\phi} + \frac{(v\cos\phi)_\phi}{r\cos\phi} + \frac{(r^2 w)_r}{r^2}.$$

The next step in the derivation is to express the state vector (u,v,w,ρ,p,Θ) as a power series expansion with respect to the small parameter, ε,

$$(16)\quad (u,v,w,\rho,p,\Theta) = (0,0,0,\rho^{(0)},p^{(0)},\Theta^{(0)}) +$$

$$+\ \varepsilon(u^{(1)},v^{(1)},w^{(1)},\rho^{(1)},p^{(1)},\Theta^{(1)}) +$$

$$+\ \varepsilon^2(u^{(2)},v^{(2)},w^{(2)},\rho^{(2)},p^{(2)},\Theta^{(2)}) + O(\varepsilon$$

The final main step is to insert the expansion (16) into the basic vector equations (1), (3), (6) which have been transformed using Eqs. (8) to (14) and equating terms having equal powers of ε in the resulting equations in the usual way. Also the equation of state (7) must be expanded and becomes

$$(17) \qquad \Theta = \Theta^{(0)} + \varepsilon\Theta^{(1)} + O(\varepsilon^2)$$

$$= \Theta^{(0)}\{1 + \varepsilon(\frac{p^{(1)}}{\gamma p^{(0)}} - \frac{\rho^{(1)}}{\rho^{(0)}}) + O(\varepsilon^2)\}$$

where $\Theta^{(0)} = (\gamma\, p_0^{\gamma-1})(p^{(0)})^{1/\gamma} / \rho^{(0)}$.

The first-order equation of state becomes

$$(18) \qquad \frac{\Theta^{(1)}}{\Theta^{(0)}} = (\frac{p^{(1)}}{\gamma p^{(0)}} - \frac{\rho^{(1)}}{\rho^{(0)}}) \ .$$

To follow the prescribed calculating routine outlined above is quite elementary so we summarize the results of the approximation procedure to $O(\varepsilon^3)$:

1) Zeroth order terms -- From the (θ,ϕ) components of the momentum equation (12) and (13) we get to lowest order in ε that $p_\theta^{(0)} = 0$ and $p_\phi^{(0)} = 0$ respectively; hence $p^{(0)}(r)$ is a function of r only, having assumed at the outset that $p_\tau^{(0)} = 0$. Then from the r-component of the momentum equation (14) to lowest order and the equation of state (17), we get $\rho^{(0)}(r)$ and consequently $\Theta^{(0)}(r)$; the atmosphere at rest with respect to the rotating earth is therefore spherically symmetric and hydrostatic,

$$(19) \qquad \frac{p_r^{(0)}}{\rho^{(0)}} + \frac{a^2 g}{r^2} = 0 \ .$$

2) First-order asymptotic equations -- From the potential vorticity equation (1) using identities (8) to (10) and the expansion (16), we get to first order,

$$\frac{d^{(1)}}{d\tau}\left\{f_3(\phi)\cdot\frac{\Theta_r^{(0)}}{\rho^{(0)}}\right\} = 0 \tag{20}$$

where

$$\frac{d^{(1)}(\;)}{d\tau} = (\;)_\tau + \frac{u^{(1)}}{r\cos\phi}(\;)_\theta + \frac{v^{(1)}}{r}(\;)_\phi + w^{(1)}(\;)_r \;. \tag{21}$$

Carrying out the differentiation explicitly, Eq. (20) becomes

$$f_2\,\frac{\Theta_r^{(0)}}{\rho^{(0)}}\,\frac{v^{(1)}}{r} + f_3\left(\frac{\Theta_r^{(0)}}{\rho^{(0)}}\right)_r w^{(1)} = 0 \tag{22}$$

where $f_2 = 2\Omega\cos\phi$ and $f_3 = 2\Omega\sin\phi$. The first term ($v^{(1)}$) in Eq. (22) is the so-called β-term. Next from the potential temperature equation (6) using the particle derivative operator (11) and the expansion (17) for we get to first order,

$$\Theta_r^{(0)}\,w^{(1)} = 0 \;. \tag{23}$$

Since the atmosphere at rest must be stable we specify that* $\Theta_r^{(0)} > 0$; otherwise the perturbation procedure is invalid. Then Eq. (23) states that

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*Possibly this condition may be relaxed in §3.2 (Perturbation on Zonal Flow).

(24) $$w^{(1)} = 0 .$$

The first order radial (vertical) velocity is zero; then Eq. (22) states that the ϕ-component of velocity $v^{(1)}$ must be zero,

(25) $$v^{(1)} = 0 .$$

Next from the Θ-component of momentum equation (12), using the particle derivative operator (11) and expansion (16), we get to first order,

(26) $$\frac{p_\Theta^{(1)}}{r\rho^{(0)}\cos\phi} + f_2 w^{(1)} - f_3 v^{(1)} = 0 .$$

But since $v^{(1)} = w^{(1)} = 0$ from Eqs. (24) and (25), this geostrophic relation degenerates to

(27) $$p_\Theta^{(1)} = 0 .$$

Thus the first-order motion is zonal with pressure $p^{(1)}(\phi,r,\tau)$. Finally to first-order, the ϕ- and r-components of the momentum equation, Eqs. (13) and (14) respectively yield,

(28) $$\frac{p_\phi^{(1)}}{r\rho^{(0)}} + f_3 u^{(1)} = 0$$

and

$$(29) \qquad \frac{1}{\rho^{(0)}}\left(p_r^{(1)} - \frac{p_r^{(0)}\rho^{(1)}}{\rho^{(0)}}\right) - f_2 u^{(1)} = 0 .$$

Eq. (28) states that the zonal flow is geostrophic; Eq. (29) becomes the hydrostatic relation only if $f_2 u^{(1)} = 0$, e.g. at the poles. Eqs. (28) and (29) are two linear equations for three unknown quantities $\rho^{(1)}(\phi,r,\tau)$, $p^{(1)}(\phi,r,\tau)$ and $u^{(1)}(\phi,r,\tau)$ assuming $\rho^{(0)}(r)$ and $p^{(0)}(r)$ are given. Thus the zonal flow is not fully determined by the first-order equations.* also we note that the first order particle derivative operator (21) operating only on zeroth and first-order terms is just the partial time derivative, $d^{(1)}(\ )/d\tau = (\ )_\tau$, since $(\ )_\theta = 0$ and $v^{(1)} = w^{(1)} = 0$ under these conditions.

3) Second-order asymptotic equations -- Following the same steps outlined in obtaining the first-order equations, the potential vorticity equation to second order yields

$$(30) \quad f_2 \frac{\Theta_r^{(0)}}{\rho^{(0)}} \frac{v^{(2)}}{r} + f_3\left(\frac{\Theta_r^{(0)}}{\rho^{(0)}}\right)_r w^{(2)} + \frac{d^{(1)}}{d\tau}\left\{\frac{1}{\rho^{(0)}}\left[\Theta_r^{(0)}\zeta^{(1)} + f_2 \frac{\Theta_\phi^{(1)}}{r} + f_3\left(\Theta_r^{(1)} - \frac{\Theta_r^{(0)}\rho^{(1)}}{\rho^{(0)}}\right)\right]\right\} = 0 .$$

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*This is exactly the same situation which we find for the linearized steady-state equations, obtained without the time scaling (cf. Eckart [7]).

Eq. (30) can be simplified since $d^{(1)}(\ )/d\tau = (\ )_\tau$ and the first order relative vorticity $\zeta^{(1)}$ is

$$(31) \qquad \zeta^{(1)} = \frac{1}{r\cos\phi}\left[v_\theta^{(1)} - (u^{(1)}\cos\phi)_\phi\right]$$

$$= -\frac{(u^{(1)}\cos\phi)_\phi}{r\cos\phi}$$

since $v^{(1)} \equiv 0$. The second-order potential temperature equation in simplified form is

$$(32) \qquad w^{(2)} + \frac{\Theta_\tau^{(1)}}{\Theta_r^{(0)}} = 0 \ .$$

Eq. (30) can be simplified by eliminating $w^{(2)}$ using Eq. (32), and the second-order potential vorticity equation becomes

$$(33) \qquad \frac{f_2 v^{(2)}}{r} + \frac{\partial}{\partial\tau}\left\{ -\frac{(u^{(1)}\cos\phi)_\phi}{r\cos\phi} + \frac{f_2\,\Theta_\phi^{(1)}}{\Theta_r^{(0)}\,r} + \right.$$

$$\left. + \frac{f_3}{\rho^{(0)}}\left[\left(\frac{\rho^{(0)}\,\Theta^{(1)}}{\Theta_r^{(0)}}\right)_r - \rho^{(1)}\right]\right\} = 0 \ .$$

Finally the second-order momentum equations are

$$(34) \qquad \frac{p_\theta^{(2)}}{r\rho^{(0)}\cos\phi} + f_2 w^{(2)} - f_3 v^{(2)} + u_\tau^{(1)} = 0$$

$$(35) \qquad \frac{1}{r\rho^{(0)}}\left(p_\phi^{(2)} - \frac{\rho^{(1)}p_\phi^{(1)}}{\rho^{(0)}}\right) + f_3 u^{(2)} + \frac{(u^{(1)})^2\tan\phi}{r} = 0$$

$$(36) \qquad \frac{1}{\rho^{(0)}}\left[p_r^{(2)} - \frac{p_r^{(0)}\rho^{(2)}}{\rho^{(0)}} + \frac{p_r^{(0)}(\rho^{(1)})^2}{(\rho^{(0)})^2} - \frac{\rho^{(1)}p_r^{(1)}}{\rho^{(0)}}\right]$$

$$- f_2 u^{(2)} - \frac{(u^{(1)})^2}{r} = 0 \ .$$

The complete second-order asymptotic equations are Eqs. (32), (33),(34), (35), and (36). The first three equations have time derivatives which operate on first-order terms only. Hence, if the first-order zonal flow is stationary, Eqs. (32), (33), and (34) state that the second-order motion is also zonal;* then the second-order terms yield no apparently useful information. Consequently in order to obtain non-trivial second-order motions, i.e. meridional planetary circulations, we must allow the first-order zonal flow to be time-dependent.** Thus we see that the first- and second-order asymptotic equations are strongly coupled to one another even though this coupled system of equations is linear. There are non-linear first-order forcing terms in the second-order equations; but from the viewpoint of the second-order equations, the first-order terms are specified. It is now evident that we can put very few, if any, additional constraints on this system of equations without drastic effects.

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* In fact, then the motion is zonal to all higher orders.

** In his careful scale analysis paper, Burger [8] emphasizes the difficulty in obtaining "specific information on the planetary scale beyond expressing its quasi-stationary character."

Now looking more closely at the second-order equations, both Eqs. (32) and (33) are relationships between single, second-order quantities and time derivatives of first-order quantities, e.g. Eq. (32) gives (differentiating with respect to longitude θ),

$$w_{\theta}^{(2)} = 0 \tag{37}$$

and Eq. (33) imposes

$$v_{\theta}^{(2)} = 0 \; . \tag{38}$$

Eqs. (37) and (38) impose the following condition on the second-order θ-momentum equation (34):

$$p_{\theta\theta}^{(2)} = 0 \; . \tag{39}$$

That is, $p^{(2)}(\phi,t,\tau)$ must be independent of θ, since Eq. (39) precludes solutions of the form $p^{(2)}(\theta,\phi,r,\tau) = p^{(2)}(\theta+2\pi,\phi,r,\tau)$; Eq. (39) states $p^{(2)}$ cannot be strictly periodic in θ.

For completeness we give the first and second order continuity equations obtained from Eq. (15) and expansion (16):

$$\rho_r^{(0)} w^{(1)} + \rho^{(0)} \operatorname{div} (u^{(1)},v^{(1)},w^{(1)}) = 0 \; . \tag{40}$$

Since $v^{(1)} = w^{(1)} = 0$ from Eqs. (24) and (25), Eq. (40) becomes $u_{\theta}^{(1)} = 0$ or $u^{(1)}(\phi,r,\tau)$ only; this gives first-order zonal flow consistent with the other first-order asymptotic equations. To second-order we have

$$\rho_r^{(0)} w^{(2)} + \rho^{(0)} \operatorname{div} (u^{(2)}, v^{(2)}, w^{(2)}) + \frac{d^{(1)}\rho^{(1)}}{d\tau} + \rho^{(1)} \operatorname{div} (u^{(1)}, v^{(1)}, w^{(1)}) = 0 \ . \tag{41}$$

This equation gives directly $u_\theta^{(2)}$ in terms of first-order quantities and is complementary to the second-order asymptotic momentum equation for $p_\theta^{(2)}$, i.e. Eq. (34). Thus the second-order continuity equation states

$$u_{\theta\theta}^{(2)} = 0 \ . \tag{42}$$

By Eqs. (37) to (42), the second-order motion must be independent of θ. To recapitulate: the first-order asymptotic motion is zonal, geostrophic, and time-dependent; the second-order asymptotic motion is also zonal, but in addition is meridional, non-geostrophic, and time-dependent.

Since now $p_\theta^{(2)} = 0$ (as well as $p_{\theta\theta}^{(2)} = 0$), the second-order terms can be eliminated among Eqs. (32), (33), and (34) and we get a first-order time-invariant,

$$(43)\quad \frac{\partial}{\partial\tau}\left\{ \frac{f_2}{r}\left(u^{(1)} - \frac{f_2\Theta^{(1)}}{\Theta_r^{(0)}}\right) - \frac{f_3(u^{(1)}\cos\phi)_\phi}{r\cos\phi} + \frac{f_2 f_3\Theta_\phi^{(1)}}{r\Theta_r^{(0)}} + \frac{f_3^2}{\rho^{(0)}}\left[\left(\frac{\rho^{(0)}\Theta^{(1)}}{\Theta_r^{(0)}}\right)_r - \rho^{(1)}\right]\right\} = 0 .$$

Eq. (43) imposes an additional constraint on the first-order solution; this condition (momentum conservation) does not seem to restrict the first-order quantities to be steady-state. However, the possible physically interesting solutions appear rather limited.

We remark that, although a numerical value for the time-scale of the fluctuating first-order motion is not determined from our analysis, the combined first- and second-order system of asymptotic equations (keeping in mind the time-scaling $\tau = \varepsilon t$) leads us to consider only slowly - time-varying motions.

4) Initial and Boundary Conditions -- In considering asymptotic motions on a planetary scale, we do not pose ordinary initial-boundary conditions as we do for moderately large-scale motions (cf. Appendix I). Instead we fully prescribe the first-order time-varying zonal motion, which acts as a forcing function on the second order asymptotic equations.

3.2. Asymptotic Equations Obtained by Perturbing on Zonal Flow, $U(\phi,r,\tau)$.

We can obtain a less-constrained* system of asymptotic equations by perturbing directly on a prescribed zonal flow, $U(\phi,r,\tau)$; in this case the first-order asymptotic equations yield time-dependent or steady state tesseral motions. Thus, we can study the low-wave-number regime in the neighborhood of zonal flow.

Since the derivation of these asymptotic equations should be clear from the detailed description of the algorithm for the simpler, but more constrained, asymptotic equations given in §3.1, we merely summarize the results here. The required changes in the procedure are the following:

1) the zeroth order solution in Eq. (16) is replaced by $(U,0,0,\rho^{(0)},p^{(0)},\Theta^{(0)})$, which are functions of (ϕ,r,τ),

2) the zeroth order absolute vorticity $(2\vec{\Omega}+\vec{\zeta})$ is $(0,f_2+\eta^{(0)},f_3+\zeta^{(0)})$ where

$$(44)\quad (\xi^{(0)},\eta^{(0)},\zeta^{(0)}) = \left\{0,\ \frac{(rU)_r}{r}\ ,\ -\ \frac{(U\cos\phi)_\phi}{r\cos\phi}\right\},$$

and

3) the particle derivative operator (11) now has a

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*In the perturbation on the atmosphere at rest, the rest state itself must be prescribed in addition to the zonal flow.

zeroth order term,

$$\frac{d(\ )}{d\tau} = \frac{U}{r\cos\phi}\ \frac{\partial(\ )}{\partial\theta} + \varepsilon\,\frac{d^{(1)}(\ )}{d\tau} + O(\varepsilon^2)$$

where the first order operator $d^{(1)}(\ )/d\tau$ is defined in Eq. (21).

The zeroth order equations, which the prescribed zonal flow $U(\phi,r,\tau)$ must satisfy, are

$$(45)\qquad p_\theta^{(0)} = 0\ , \qquad\qquad (\text{ zonal })$$

$$(46)\qquad \frac{U^2\tan\phi}{r} + \frac{p_\phi^{(0)}}{r\rho^{(0)}} + f_3U = 0\ , \qquad (\text{"geostrophic"})$$

$$(47)\qquad -\frac{U^2}{r} + \frac{p_r^{(0)}}{\rho^{(0)}} + \frac{ga^2}{r^2} - f_2U = 0\ , \qquad (\text{"hydrostatic"})$$

These zeroth order equations (45,46,47) are comparable to the first order asymptotic equations (27,28,29).

The first order asymptotic equations which follow are the principal approximation equations. From the potential vorticity equation (1) we get

$$(48)\qquad \frac{U}{r\rho^{(0)}\cos\phi}\ \frac{\partial}{\partial\theta}\left\{\left(\frac{\Theta_\phi^{(0)}}{r}\,\eta^{(1)} + \Theta_r^{(0)}\zeta^{(1)}\right) + \right.$$

$$\left. + \frac{(f_2+\eta^{(0)})}{r}\left(\Theta_\phi^{(1)} - \frac{\Theta_\phi^{(0)}}{\rho^{(0)}}\,\rho^{(1)}\right) + \right.$$

$$+ (f_3 + \zeta^{(0)}) \left(\Theta_r^{(1)} - \frac{\Theta_r^{(0)} \rho^{(1)}}{\rho^{(0)}} \right) \Bigg\} +$$

$$+ \frac{d_m^{(1)}}{d\tau} \left\{ \frac{1}{\rho^{(0)}} \left[(f_2 + \eta^{(0)}) \frac{\Theta_\phi^{(0)}}{r} + (f_3 + \zeta^{(0)}) \Theta_r^{(0)} \right] \right\} = 0$$

where

$$\frac{d_m^{(1)}(\ )}{d\tau} = (\ )_\tau + \frac{v^{(1)}}{r} (\ )_\phi + w^{(1)} (\ )_r \ . \tag{49}$$

From the potential temperature equation we get

$$\frac{U}{r \cos \phi} \Theta_\theta^{(1)} + \frac{d_m^{(1)} \Theta^{(0)}}{d\tau} = 0 \ . \tag{50}$$

From the (θ, ϕ, r) momentum equations respectively we obtain

$$\frac{U}{r} \left(\frac{u_\theta^{(1)}}{\cos \phi} + w^{(1)} - v^{(1)} \tan \phi \right) + \frac{p_\theta^{(1)}}{r \rho^{(0)} \cos \phi} + \tag{51}$$

$$+ f_2 w^{(1)} - f_3 v^{(1)} = 0$$

$$\frac{2Uu^{(1)} \tan \phi}{r} + \frac{1}{r \rho^{(0)}} \left(p_\phi^{(1)} - \frac{p_\phi^{(0)} \rho^{(1)}}{\rho^{(0)}} \right) + f_3 u^{(1)} = 0 \tag{52}$$

$$- \frac{2Uu^{(1)}}{r} + \frac{1}{\rho^{(0)}} \left(p_r^{(1)} - \frac{p_r^{(0)} \rho^{(1)}}{\rho^{(0)}} \right) - f_2 u^{(1)} = 0 \ . \tag{53}$$

The first order equations (48,50,51,52,53) are comparable to the second order equations (33,32,34,35,36). Eq. (51) states that the first order meridional motion is non-geostrophic. Only equations (48,50) have time derivatives; and the general structure of these first order asymptotic equations are rather similar to the second order asymptotic equations in §3.1. However, these first order equations exhibit a coupling between the meridional motion and the longitudinal (θ) variation (or zonal wave number). In this asymptotic formulation the zeroth order zonal transport of the first order potential vorticity and potential temperature is directly related (proportional) to the first order meridional transport of the zeroth order potential vorticity and potential temperature respectively. Non-stationary planetary-scale wave motions occur only with unsteady zonal motion. Again as in §3.1 a thorough numerical study of these asymptotic equations will be necessary in order to test their validity for describing either steady or slowly-varying planetary-scale motions of our atmosphere.

Appendix I. The Tangent-Plane Approximation for Adiabatic Flow.

The tangent-plane approximation is easily obtained from the derivation outlined in §3.1; for this special case, we get $f_2 = 2\Omega\cos\phi = \text{constant} = f_{2_0}$ and $f_3 = 2\Omega\sin\phi = \text{constant} = f_{3_0}$. The zeroth order terms are unaffected.

From the potential vorticity equation for first- and second-order, respectively, we get

$$(\text{I-1}) \qquad f_{3_0}\left(\frac{\Theta_r^{(0)}}{\rho^{(0)}}\right)_r w^{(1)} = 0$$

instead of Eq. (22) and

$$(\text{I-2}) \quad f_{3_0}\left(\frac{\Theta_r^{(0)}}{\rho^{(0)}}\right)_r w^{(2)} + \frac{d^{(1)}}{d\tau}\left\{\frac{1}{\rho^{(0)}}\left[\Theta_r^{(0)}\zeta^{(1)} + f_{3_0}\left(\Theta_r^{(1)} - \frac{\Theta_r^{(0)}\rho^{(1)}}{\rho^{(0)}}\right)\right]\right\} = 0$$

where

$$\frac{d^{(1)}(\ )}{d\tau} = (\ )_\tau + \frac{u^{(1)}}{r\cos\phi}(\ )_\theta + \frac{v^{(1)}}{r}(\ )_\phi$$

is the horizontal particle derivative operator since $w^{(1)} = 0$; and

$$\zeta^{(1)} = \frac{1}{r\cos\phi}\left[v_\theta^{(1)} - (u^{(1)}\cos\phi)_\phi\right]$$

instead of Eqs. (30), (21) and (31), respectively.

From the potential temperature equation for first- and second-order, respectively, we obtain the same equations as for the planetary case, namely Eqs. (23) and (32). Both Eqs. (I-1) and (23) state that $w^{(1)} = 0$ for $\Theta_r^{(0)} > 0$; and from the second order Eqs. (I-2) and (32) we can eliminate the radial velocity $w^{(2)}$ and get a modified (by the $d\Theta^{(1)}/d\tau$ term) potential vorticity conservation equation of only first order quantities;

$$\text{(I-3)} \qquad \frac{d^{(1)}}{d\tau}\{\zeta^{(1)} + \frac{f_{3_0}}{\rho^{(0)}}[(\frac{\rho^{(0)}\Theta^{(1)}}{\Theta_r^{(0)}})_r - \rho^{(1)}] = 0 .$$

The first order momentum equations are

$$\text{(I-4)} \qquad \frac{p_\theta^{(1)}}{\rho^{(0)}\, r\cos\phi} - f_{3_0} v^{(1)} = 0 \qquad \text{(geostrophic)}$$

$$\text{(I-5)} \qquad \frac{p_\phi^{(1)}}{\rho^{(0)}\, r} + f_{3_0} u^{(1)} = 0 \qquad \text{(geostrophic)}$$

$$\text{(I-6)} \qquad p_r^{(1)} + \frac{ga^2}{r^2}\rho^{(1)} = 0$$

We note that the ϕ-component of the coriolis parameter f_{2_0} does not appear in these first order equations.

By introducing a new variable $\Psi \equiv \frac{p^{(1)}}{f_{3_0}}$, the momentum stream function, and introducing cartesian coordinates by the transformations $(\ )_\theta/r\cos\phi = (\ )_x$,

$(\ )_\phi/r = (\ )_y$, $(\ )_r = (\ )_z$ and $ga^2/r^2 \approx g$, the Eqs. (I-3) to (I-6), inclusive, become in more natural coordinates for the plane case,

$$\frac{d^{(1)}}{d\tau}\left\{\Delta\Psi + \frac{f_{3_o}^2}{g}\left[\left(\frac{\Psi_z + \frac{g}{(c^{(0)})^2}\Psi}{S^{(0)}}\right)_z + \Psi_z\right]\right\} = 0 \tag{I-7}$$

$$\rho^{(0)}(u^{(1)},\ v^{(1)}) = (-\Psi_y, \Psi_x) \qquad \text{(geostrophic)} \tag{I-8}$$

$$\rho^{(1)} = \frac{f_{3_o}}{g}\Psi_z \tag{I-9}$$

where

$$\frac{d^{(1)}}{d\tau} = (\ )_\tau + u^{(1)}(\ )_x + v^{(1)}(\ )_y \tag{I-10}$$

$$\begin{aligned}\Delta\Psi &= \Psi_{xx} + \Psi_{yy} \\ &= \rho^{(0)}\zeta^{(1)} = \rho^{(0)}(v_x^{(1)} - u_y^{(1)}))\end{aligned} \tag{I-11}$$

$$S^{(0)} = \frac{\Theta_z^{(0)}}{\Theta^{(0)}} = -\left(\frac{\rho_z^{(0)}}{\rho^{(0)}} + \frac{g}{(c^{(0)})^2}\right) > 0 \tag{I-12}$$

$$(c^{(0)})^2 = \gamma R T^{(0)} = \frac{\gamma p^{(0)}}{\rho^{(0)}} \tag{I-13}$$

in which $S^{(0)}$ is the static stability parameter, $c^{(0)}$ is the velocity of sound, R is the gas constant, and $T^{(0)}$ is the static temperature; $T^{(0)}$ is a given function of height z.

The initial condition for the system of equations (I-7) and (I-8) is: prescribe $\Psi(x,y,z,o)$ initially. The boundary conditions are $\rho w = 0$ at $z = 0,\infty$; to $O(\varepsilon^2)$ this condition becomes $\rho^{(0)} w^{(2)} = 0$ or by Eq. (32)

$$(\text{I-14}) \qquad \frac{d^{(1)}}{d\tau} \left(\Psi_z + \frac{g}{(c^{(0)})^2} \Psi \right) = 0$$

for $z = 0,\infty$.

Presumably the geostrophic potential vorticity conservation equation (I-7) describes slowly varying, moderately large-scale motions in the atmosphere away from the equator. Again as in the main text of this paper, numerical estimates on how slowly varying and what horizontal length scales are involved is missing from this analysis. Some information about these important requirements is given in Appendix II. We mention that we have not been able to include the β-term (cf. Eq.(22)) in Eq. (I-7) in a consistent manner. To calculate the second order, non-geostrophic system of linear equations, similar to that derived in Ref. [2], is quite straightforward.

Appendix II. Vertical Structure of a Rectilinear Geostrophic Vortex Line Element in Adiabatic Flow; Relationship between the Strength, Horizontal Length Scale, and the Static-Atmospheric Distribution.

The vortex line element is obtained by seeking singular solutions of the geostrophic potential vorticity conservation equation (I-7). This concept is strictly valid in the tangent-plane approximation -- but also to good approximation, we feel, in application to flows in middle latitudes so long as the motion of these vortex line elements does not deviate latitudinally very far from the initial latitude position. We assume steady-state motion and seek solutions of

$$\text{(II-1)} \qquad \{\Delta\Psi + \frac{f_{3_0}}{g}[(\frac{\Psi_z + \frac{g}{(c^{(0)})^2}\Psi}{S^{(0)}})_z + \Psi_z]\} = \delta(r)\cdot\Gamma(z)$$

where $\delta(r)$ is the delta function of $r = (x^2+y^2)^{1/2}$, i.e. $\delta(r)$ is singular at $r = 0$ and its integral over the entire x-y plane is equal to a constant, say one; $\Gamma(z)$ is the strength of the vortex line element, and is to be determined. Since we assume that the only flow in the entire three-dimensional space derives from Eq. (II-1), then a potential temperature invariant also exists

$$\text{(II-2)} \qquad (\Psi_z + \frac{g}{(c^{(0)})^2}\Psi)_\tau = 0 \ .$$

We note that in time-dependent adiabatic flows, the potential temperature is advected with the *horizontal* flow only at the boundaries, e.g. the ground (cf. Appendix I). The time derivative of Eq. (II-1), which is also a time invariant, combined with Eq. (II-2), implies that we are seeking solutions of a simpler equation than Eq. (II-1), namely

$$\text{(II-3)} \qquad (\Delta\Psi + \frac{f_{3_0}^2}{g}\,\Psi_z) = \delta(r)\cdot\Gamma(z)$$

or using Eq. (II-2) again,

$$\text{(II-4)} \qquad (\Delta\Psi - \frac{f_{3_0}^2}{(c^{(0)})^2}\,\Psi) = \delta(r)\;\;\Gamma(z)\ .$$

Since Eq. (II-4) only has horizontal derivatives in the x- and y-directions (not z), there is an axially-symmetric singular solution of Eq. (II-4) of the form,

$$\text{(II-5)} \qquad \Psi_0 = \Gamma(z)\;\; K_0[\kappa(z)\cdot r]$$

where $\kappa(z)$ is given by

$$\text{(II-6)} \qquad \kappa(z) = \frac{f_{3_0}}{c^{(0)}(z)}$$

and $K_0(\kappa r)$ is the modified Bessel function of the second kind. We note that solution (II-5) is *not a "separation-of-variables"-type solution* of Eq.(II-4).

A functional form for $\Gamma(z)$ can be found by replacing Eq. (II-2) by

$$\text{(II-7)} \qquad \Psi_z + \frac{g}{(c^{(0)})^2}\,\Psi = \frac{\partial K_0}{\partial z}\cdot\Gamma(z) \ .$$

We note that the right-hand side of Eq. (II-7) is regular in r. Then it is easily seen that $\Gamma(z)$ may take the form

$$\text{(II-8)} \qquad \Gamma(z) = \Gamma_0 \exp\left\{- g\int_0^z \frac{d\sigma}{[c^{(0)}(\sigma)]^2}\right\}$$

where Γ_0 = constant; and the solution (II-5) of Eq. (II-4) also satisfies Eq. (II-7). Hence we can write solution (II-5) more explicitly as

$$\text{(II-9)} \quad \Psi_0(r,z) = \Gamma_0\cdot K_0[\kappa(z)\cdot r]\ \exp\left\{- g\int_0^z \frac{d\sigma}{[c^{(0)}(\sigma)]^2}\right\}$$

where $\kappa(z)$ is given by Eq. (II-6). Solution (II-9) relates both $\Gamma(z)$ and $\kappa(z)$ to the given static atmosphere; and therefore there is a functional relationship between the strength $\Gamma(z)$ and the horizontal length scale $\kappa^{-1}(z)$, aside from an arbitrary constant Γ_0. From Eqs. (II-6) and (II-8) this relationship is

$$\text{(II-10)} \qquad \frac{\Gamma(z)}{\Gamma_0} = \exp\left\{\frac{-g}{f_{3_0}^2}\int_0^z \kappa^2(\sigma)\ d\sigma\right\} ;$$

or, differentiating with respect to z,

$$\text{(II-11)} \qquad \frac{1}{\Gamma(z)} \frac{d\Gamma(z)}{dz} = \frac{d[\ln \Gamma(z)]}{dz} = - \frac{g}{f^2_{3_0}} \kappa^2(z)$$

which eliminates the integration constant Γ_0. Since the right hand side of (II-11) is definitely negative, $d\Gamma(z)/d(z)$ must be negative when friction can be ignored, that is, not close to the ground and possibly not extremely high up. We can get a somewhat simpler expression for $\Gamma(z)$ than Eq. (II-10), using the hydrostatic condition which holds for the static atmosphere,

$$\text{(II-12)} \qquad \ln \left[\frac{\Gamma(z)}{\Gamma_0}\right] = \frac{1}{\gamma} \ln \left[\frac{p^{(0)}}{p_0}\right]$$

where $\gamma = C_p/C_v$ and p_0 is the pressure at the ground, say. Eq. (II-12) states that

$$\text{(II-13)} \qquad \Gamma(z) = (\text{constant}) \cdot [p^{(0)}(z)]^{1/\gamma}$$

$$= (\text{constant}) \cdot \rho^{(0)}(z) \cdot \Theta^{(0)}(z) \ .$$

That is, the strength $\Gamma(z)$ is proportional to the product of the static density and static potential temperature.

In addition, we note that there is an identity (for fixed z) which serves to define the strength $\Gamma(z)$,

$$(\text{II-14}) \qquad \Gamma(z) = \oint_C \frac{\partial \Psi_0}{\partial n}\, ds - \kappa^2(z) \iint_R \Psi_0 \, dx\, dy$$

which might be useful for calculating $\Gamma(z)$ from actual data for mature hurricanes and typhoons (cf. Refs. [2] and [9]); the calculated results can be compared to Eqs. (II-10), (II-11), or (II-13). In Eq. (II-14) C is an arbitrary horizontal closed contour, surrounding the hurricane and enclosing the area R.

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Asymptotic equations describing slowly varying planetary-scale motions of the atmosphere over the rotating earth are derived and compared to the geostrophic equations of motion in the tangent plane approximation. Since the same approximation procedure is used in both the spherical and plane geometry, a direct term by term comparison is possible. The plane approximation yields familiar prognostic equations (initial value problem); but in the more constrained spherical geometry, a more restricted forcing-type problem results if the zonal motion is time-dependent. That is, with a given slowly varying or steady zonal motion, is associated a class of long-wave tesseral motions, i.e. coupled low-wave-number and meridional motions.

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